The Uniformizability of L-topological Groups

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Abstract:
In this paper, we show that any stratified L-topological group (G,τ) is uniformizable. That is, we define, using the family of prefilters which corresponds the fuzzy neighborhood filter at the identity element of (G,τ), unique left and right invariant fuzzy uniform structures on G compatible with the fuzzy topology τ. On the other hand, on any group G, using a family of prefilters on G fulfills certain conditions, we construct those left and right fuzzy uniform structures which induce a stratified fuzzy topology τ on G for which (G,τ) is a stratified L-topological group and this family of prefilters coincides with the family of prefilters corresponding to the fuzzy neighborhood filter at the identity element of (G,τ). Moreover, we show the relation between the L-topological groups and the GT_i-spaces, such as: the fuzzy topology of an L-topological group (resp., a separated L-topological group) is completely regular, (resp., $GT_{3\frac{1}{2}}$)

Keywords:
Fuzzy filters; Fuzzy uniform spaces; Fuzzy topological groups; GT_i-spaces; Completely regular spaces; $GT_{3\frac{1}{2}}$-spaces; L-Tychonoff spaces.

1. Introduction

The notion of an L-topological group (G,τ) is defined by Ahsanullah [1] in 1984 as an ordinary group G equipped with a fuzzy topology τ on G such that the binary operation and the unary operation of the inverse are fuzzy continuous with respect to τ. In [1,7], many results on the L-topological groups are studied. These L-topological

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groups are called, in [1], fuzzy topological groups. An L-topological group (G,τ) is called stratified if the L-topology τ is stratified.

The fuzzy neighborhood filter at the identity element of the stratified L-topological group (G,τ) corresponds a family of prefilters on G [11]. Using this family of prefilters, we construct, in this paper, a unique left invariant fuzzy uniform structure $\mathcal{U}^l$ and a unique right invariant fuzzy uniform structure $\mathcal{U}^r$ on G. These fuzzy uniform structures $\mathcal{U}^l$ and $\mathcal{U}^r$ are compatible with τ, that is, $\tau_{\mathcal{U}^l} = \tau_{\mathcal{U}^r} = \tau$. This means that the stratified L-topological group (G,τ) is uniformizable. The fuzzy uniform structures $\mathcal{U}^l$ and $\mathcal{U}^r$ are fuzzy uniform structures in sense of [12] which are defined as fuzzy filters on the cartesian product $G\times G$ of G with itself. We show also here that for any group G and any family of prefilters fulfills certain conditions, we define the left and the right fuzzy uniform structures $\mathcal{U}^l$ and $\mathcal{U}^r$ on G such that $\tau_{\mathcal{U}^l} = \tau_{\mathcal{U}^r}$ is a stratified fuzzy topology τ on G for which the pair (G,τ) is a stratified L-topological group. Moreover, this family of prefilters is exactly the family of prefilters which corresponds the fuzzy neighborhood filter at the identity element of the stratified L-topological group (G,τ).

Moreover, in this paper, we study some relations between the L-topological groups and the fuzzy separation axioms GT_i which we had introduced in [2,3,5]. We show that the fuzzy topology τ of an L-topological group (G,τ) is completely regular in our sense [5] and that the L-topological group (G,τ) is separated if and only if the fuzzy topology τ is GT_0 (resp. GT_1, GT_2, $GT_{3\frac{1}{2}}$) if and only if the left fuzzy uniform structure $\mathcal{U}^l$ (resp. the right fuzzy uniform structure $\mathcal{U}^r$) is separated.

2. On fuzzy filters

Let L be a complete chain with different least and greatest elements 0 and 1, respectively. Let $L_0 = L\setminus\{0\}$ and $L_1 = L\setminus\{1\}$. Denote by L^X the set of all fuzzy subsets of a non-empty set X. By a *fuzzy filter* on X [9,10] is meant a mapping $\mathcal{M}: L^X \to L$ such that $\mathcal{M}(\overline{\alpha}) \le \alpha$ holds for all $\alpha \in L$ and $\mathcal{M}(\overline{1}) = 1$, and also $\mathcal{M}(f \wedge g) = \mathcal{M}(f) \wedge \mathcal{M}(g)$ for all $f, g \in L^X$. A fuzzy filter $\mathcal{M}$ is called *homogeneous* if $\mathcal{M}(\overline{\alpha}) = \alpha$ for all $\alpha \in L$. If $\mathcal{M}$ and $\mathcal{N}$ are fuzzy filters on X, $\mathcal{M}$ is said to be *finer than* $\mathcal{N}$, denoted by, $\mathcal{M} \le \mathcal{N}$, provided $\mathcal{M}(f) \ge \mathcal{N}(f)$ holds for all $f \in L^X$. By $\mathcal{M} \not\le \mathcal{N}$ we denote that $\mathcal{M}$ is not finer than $\mathcal{N}$.

For any set A of fuzzy filters on X, the infimum $\bigwedge_{\mathcal{M}\in A} \mathcal{M}$, with respect to the finer relation on fuzzy filters, does not exist in general. The infimum $\bigwedge_{\mathcal{M}\in A} \mathcal{M}$ of A exists *if and only if* for each non-empty finite subset $\{\mathcal{M}_1, \cdots, \mathcal{M}_n\}$ of A we have

$\mathcal{M}_1(f_1)\wedge\cdots\wedge\mathcal{M}_n(f_n)\le\sup(f_1\wedge\cdots\wedge f_n)$ for all $f_1,\cdots,f_n\in L^X$ [9]. If the infimum of A exists, then for each $f\in L^X$ and n as a positive integer we have

$$\Big(\bigwedge_{\mathcal{M}\in A}\mathcal{M}\Big)(f)=\bigvee_{\substack{f_1\wedge\cdots\wedge f_n\le f,\\ \mathcal{M}_1,\cdots,\mathcal{M}_n\in A}}\big(\mathcal{M}_1(f_1)\wedge\cdots\wedge\mathcal{M}_n(f_n)\big).$$

A *prefilter* on X is a non-empty subset $\mathcal{F}$ of L^X which does not contain $\overline{0}$ and closed under finite infima and super sets [15]. For each fuzzy filter $\mathcal{M}$ on X, the subset α-pr $\mathcal{M}$ of L^X defined by:

$$\alpha-\mathrm{pr}\,\mathcal{M}=\left\{f\in L^X\,\middle|\,\mathcal{M}(f)\ge\alpha\right\}$$

is a prefilter on X.

A *valued fuzzy filter base* on a set X [10] is a family $(\mathcal{B}_\alpha)_{\alpha\in L_0}$ of non-empty subsets of L^X such that the following conditions are fulfilled:

(V1) $f\in\mathcal{B}_\alpha$ implies $\alpha\le\sup f$.

(V2) For al $\alpha,\beta\in L_0$ and all mappings $f\in\mathcal{B}_\alpha$ and $g\in\mathcal{B}_\beta$, if even $\alpha\wedge\beta>0$ holds, then there are a $\gamma\ge\alpha\wedge\beta$ and a fuzzy set $h\le f\wedge g$ such that $h\in\mathcal{B}_\gamma$.

Each valued fuzzy filter base $(\mathcal{B}_\alpha)_{\alpha\in L_0}$ on a set X defines a fuzzy filter $\mathcal{M}$ on X by $\mathcal{M}(f)=\bigvee_{g\in\mathcal{B}_\alpha,g\le f}\alpha$ for all $f\in L^X$. On the other hand, each fuzzy filter $\mathcal{M}$ can be generated by many valued fuzzy filter bases, and among them the greatest one $(\alpha-\mathrm{pr}\mathcal{M})_{\alpha\in L_0}$.

Proposition 2.1 [10]. *There is a one-to-one correspondence between the fuzzy filters* $\mathcal{M}$ *on* X *and the families* $(\mathcal{M}_\alpha)_{\alpha\in L_0}$ *of prefilters on* X *which fulfill the following conditions:*

(1) $f\in\mathcal{M}_\alpha$ *implies* $\alpha\le\sup f$.

(2) $0<\alpha\le\beta$ *implies* $\mathcal{M}_\alpha\supseteq\mathcal{M}_\beta$.

(3) For *each* $\alpha\in L_0$ *with* $\bigvee_{0<\beta<\alpha}\beta=\alpha$ *we have* $\bigcap_{0<\beta<\alpha}\mathcal{M}_\beta=\mathcal{M}_\alpha$.

This correspondence is given by $\mathcal{M}_\alpha=\alpha-\mathrm{pr}\mathcal{M}$ *for all* $\alpha\in L_0$ *and* $\mathcal{M}(f)=\bigvee_{g\in\mathcal{M}_\alpha,g\le f}\alpha$ *for all* $f\in L^X$.

Fuzzy neighborhood filters. In the following the fuzzy topology τ on a set X in sense of [8,13] will be used, int_τ and cl_τ denote the interior and the closure operators with respect to τ, respectively. For each fuzzy topological space (X,τ) and each $x\in X$ the mapping $\mathcal{N}(x):L^X\to L$ defined by

$$\mathcal{N}(x)(f)=\mathrm{int}_\tau f(x)$$

for all $f \in L^X$ is a fuzzy filter on X, called the *fuzzy neighborhood filter* of the space (X,τ) at x [11].

$f \in L^X$ is called a τ *-neighborhood* at $x \in X$ provided $\alpha \le \operatorname{int}_\tau f(x)$ for some $\alpha \in L_0$. That is, f is a τ-neighborhood at x if $f \in \alpha - \operatorname{pr}\mathcal{N}(x)$ for some $\alpha \in L_0$.

Let (X,τ) and (Y,σ) be two fuzzy topological spaces. Then the mapping $f:(X,\tau) \to (Y,\sigma)$ is called *fuzzy continuous* (or (τ,σ) *-continuous*) provided $\operatorname{int}_\sigma g \circ f \le \operatorname{int}_\tau (g \circ f)$ for all $g \in L^Y$.

3. L topological groups

In the following we focus our study on a multiplicative group G. We denote, as usual, the identity element of G by e and the inverse of an element a of G by a^{-1}.

Let $\pi : G \times G \to G$ be a mapping defined by

$$\pi(a,b) = ab \text{ for all } a,b \in G,$$

and $i : G \to G$ a mapping defined by

$$i(a) = a^{-1} \text{ for all } a \in G,$$

that is, π and i are the binary operation and the unary operation of the inverse on G, respectively.

Here, we define the product of $f,g \in L^G$ with respect to the binary operation π on G as the fuzzy set fg in G defined by:

$$fg = \bigwedge_{f(x)>0,\, g(y)>0} (xy)_1 \tag{3.1}$$

In particular, for all $a \in G$ and all $f \in L^G$, we have $af \in L^G$ defined by

$$af = \bigwedge_{f(x)>0} (ax)_1 \tag{3.2}$$

and $fa \in L^G$ defined by

$$fa = \bigwedge_{f(x)>0} (xa)_1 . \tag{3.3}$$

Also, we can define the inverse of $f \in L^G$ with respect to the unary operation i on G as the fuzzy set f^{-1} on G by:

$$f^{-1}(x) = f(x^{-1}) \text{ for all } x \in G. \tag{3.4}$$

The following definitions are similar to those in [14].

Definition 3.1. Let τ be a fuzzy topology on a group G. The mapping $\pi:(G\times G,\tau\times\tau)\to(G,\tau)$ is called $(\tau\times\tau,\tau)$-*continuous in each variable separately* if for all $f\in\alpha-\mathrm{pr}\mathcal{N}(ab)$, there exists $g\in\alpha-\mathrm{pr}\mathcal{N}(b)$ such that $ag\le f$ or there exists $h\in\alpha-\mathrm{pr}\mathcal{N}(a)$ such that $hb\le f$ for some $\alpha\in L_0$ and for all $a,b\in G$.

Definition 3.2. Let G be a group and τ be a fuzzy topology on G. Then the pair (G,τ) will be called a *semi-* L *-topological group* if the mapping π is $(\tau\times\tau,\tau)$-continuous in each variable separately.

Definition 3.3. The mapping π is called $(\tau\times\tau,\tau)$-*continuous everywhere* if for all $f\in\alpha-\mathrm{pr}\mathcal{N}(ab)$, there exist $g\in\alpha-\mathrm{pr}\mathcal{N}(a)$ and $h\in\alpha-\mathrm{pr}\mathcal{N}(b)$ such that $gh\le f$ for some $\alpha\in L_0$ and for all $a,b\in G$.

Definition 3.4. The mapping i is called (τ,τ) -*continuous* if for all $f\in\alpha-\mathrm{pr}\mathcal{N}(a^{-1})$, there exists an $g\in\alpha-\mathrm{pr}\mathcal{N}(a)$ such that $g^{-1}\le f$ for some $\alpha\in L_0$ and for all $a\in G$.

Definition 3.5 [1]. Let G be a group and τ be a fuzzy topology on X. Then the pair (G,τ) will be called an L *-topological group* if the mapping π is $(\tau\times\tau,\tau)$-continuous everywhere and the mapping i is (τ,τ)-continuous.

Clearly, every L -topological group is a semi- L -topological group.

Proposition 3.1. *The pair* (G,τ) *is an* L *-topological group if and only if for all* $f\in\alpha-\mathrm{pr}\mathcal{N}(a^{-1}b)$, *there exist* $g\in\alpha-\mathrm{pr}\mathcal{N}(a)$ *and* $h\in\alpha-\mathrm{pr}\mathcal{N}(b)$ *such that* $g^{-1}h\le f$ *for some* $\alpha\in L_0$ *and for all* $a,b\in G$.

Proof. Obvious.

Let us call a fuzzy set $f\in L^G$ symmetric if the inverse f^{-1}, defined by (3.4), fulfills that $f=f^{-1}$. For each group G and $a\in G$, the *left* and *right translations* are the homomorphism $l_a:G\to G$ defined by $l_a(x)=ax$ and $R_a:G\to G$ defined by $R_a(x)=xa$ for each $x\in G$, respectively. The left and right translations in L -topological groups fulfill the following result.

Proposition 3.2 [7]. *Let* (G,τ) *be an* L *-topological group. Then for each* $a\in G$ *the left and right translations* l_a *and* R_a *are* L *-homeomorphisms.*

We shall use the following result.

Lemma 3.1. *Let f be an open fuzzy set in an L-topological group (G,τ). Then for any $x_0 \in G$ the fuzzy sets fx_0 and $x_0 f$ are also open.*

Proof. Consider the mapping

$$h : G \to G \times G,\ x \mapsto \left(x_0^{-1}, x\right)$$

and the projection mappings

$$p_1 : G \times G \to G,\ (x_1, x_2) \mapsto x_1$$

and

$$p_2 : G \times G \to G,\ (x_1, x_2) \mapsto x_2 .$$

Then $(p_1 \circ h)(x) = x_0^{-1}$ and $(p_2 \circ h)(x) = x$. Since $(p_1 \circ b)$ and $(p_2 \circ h)$ are (τ,τ)-continuous, then h is also $(\tau, \tau\times\tau)$-continuous. Now, we have

$$\pi : G \times G \to G,\ (x_1, x_2) \mapsto x_1 x_2$$

is $(\tau\times\tau, \tau)$-continuous, and thus the mapping $\lambda = \pi \circ h$, for which $\lambda(x) = \pi(h(x)) = \pi\left(x_0^{-1}, x\right) = x_0^{-1}x$ for all $x \in G$, is (τ,τ)-continuous. Also, $\lambda^{-1}\left(x_0^{-1}x\right) = x$ for all $x \in G$, that is, $\lambda^{-1}(x) = x_0 x$ for all $x \in G$. In particular, $x_0 f = \lambda^{-1}(f)$ is a fuzzy open set in (G,τ). fx_0 is also open with a similar proof.

Recall that: If $f : X \to Y$ is a mapping between the non-empty sets X and Y and $h \in L^Y$, then the *preimage* $f^{-1}(h)$ of h with respect to f is defined by $f^{-1}(h) = h \circ f$.

Now, we prove the following result.

Lemma 3.2.*Let (G,τ) be an L-topological group and $x_0 \in G$.Then $f \in \alpha - \mathrm{pr}\mathcal{N}(e)$ if and only if $x_0 f \in \alpha - \mathrm{pr}\mathcal{N}(x_0)$ if and only if $fx_0 \in \alpha - \mathrm{pr}\mathcal{N}(x_0)$.*

Proof. Since the mapping $\lambda = \pi \circ h$, as in Lemma 3.1, is (τ,τ)-continuous, then $\mathrm{int}_\tau\, g \circ \lambda \le \mathrm{int}_\tau (g \circ \lambda)$ for all $g \in L^G$. That is,

$$\mathrm{int}_\tau\, f\left(x_0^{-1}x\right) = \mathrm{int}_\tau\, f(\lambda(x)) \le \mathrm{int}_\tau (f \circ \lambda)(x) = \mathrm{int}_\tau \left(\lambda^{-1}(f)\right)(x) = \mathrm{int}_\tau (x_0 f)(x)$$

for all $x \in G$ and all $f \in L^G$. Hence, $f \in \alpha - \mathrm{pr}\mathcal{N}(e)$ if and only if $x_0 f \in \alpha - \mathrm{pr}\mathcal{N}(x_0)$. The other case is similar and the proof is then complete.

4. L-topological groups and their canonical fuzzy uniform structures

An L-topological group (G,τ) is called *stratified* if the L-topology τ is stratified, that is, all constant fuzzy sets $\bar{\alpha}$ belong to τ. In the sequel we show that for each stratified L-topological group (G,τ), there are unique left and right invariant fuzzy uniform structures on G compatible with τ.

For a family $(\mathcal{V}_\alpha)_{\alpha\in L_0}$ of subsets $\mathcal{V}_\alpha$ of L^X, consider the following conditions:

(e1) For all $\alpha \in L_0$, if $0<\beta\le\alpha$, then $\mathcal{V}_\alpha \subseteq \mathcal{V}_\beta$,

(e2) For all $\alpha \in L_0$ with $\bigvee_{0<\beta<\alpha}\beta=\alpha$, we have $\mathcal{V}_\alpha = \bigcap_{0<\beta<\alpha}\mathcal{V}_\beta$,

(e3) For all $\alpha \in L_0$ and all $f\in\mathcal{V}_\alpha$, we have $\alpha \le \sup f$,

(e4) For all $\alpha \in L_0$ and all $f\in\mathcal{V}_\alpha$, there exists $g\in\mathcal{V}_\alpha$ such that $g^{-1}\le f$,

(e5) For all $\alpha \in L_0$ and all $f\in\mathcal{V}_\alpha$, there exists $g\in\mathcal{V}_\alpha$ such that $gg\le f$.

Proposition 4.1. *Let $\mathcal{N}(e)$ be the fuzzy neighborhood filter at the identity element e of an L-topological group (G,τ). Then the family $(\alpha-\mathrm{pr}\mathcal{N}(e))_{\alpha\in L_0}$ of prefilters $\alpha-\mathrm{pr}\mathcal{N}(e)$ fulfills the conditions (e1) – (e5).*

Proof. Since $0<\beta\le\alpha$ and $f\in\alpha-\mathrm{pr}\mathcal{N}(e)$ imply that $\beta\le\alpha\le\mathrm{int}_\tau f(e)$, then $f\in\beta-\mathrm{pr}\mathcal{N}(e)$. Hence, $\alpha-\mathrm{pr}\mathcal{N}(e)\in\beta-\mathrm{pr}\mathcal{N}(e)$, and (e1) is fulfilled.

From (e1), we get that $\alpha-\mathrm{pr}\mathcal{N}(e)\subseteq\bigcap_{0<\beta<\alpha}\beta-\mathrm{pr}\mathcal{N}(e)$. Now, if $f\in\bigcap_{0<\beta<\alpha}\beta-\mathrm{pr}\mathcal{N}(e)$, then $f\in\beta-\mathrm{pr}\mathcal{N}(e)$ for all $\beta\in L_0$ with $\alpha=\bigvee_{0<\beta<\alpha}\beta$, which means that $f\in\alpha-\mathrm{pr}\mathcal{N}(e)$ and hence (e2) holds.

(e3) is evident.

Since $i(e)=e^{-1}=e$ and i is (τ,τ)-continuous, then (e4) is fulfilled.

Since $\pi(e,e)=ee=e$ and π is $(\tau\times\tau,\tau)$-continuous every where, then (e5) is fulfilled.

Fuzzy uniform structures. Let $\mathcal{U}$ be a fuzzy filter on $X\times X$. The *inverse* $\mathcal{U}^{-1}$ of $\mathcal{U}$ is a fuzzy filter on $X\times X$ defined by $\mathcal{U}^{-1}(u)=\mathcal{U}(u^{-1})$ for all $u\in L^{X\times X}$, where u^{-1} is the inverse of u defined by: $u^{-1}(x,y)=u(y,x)$ for all $x,y\in X$. Let, each $\alpha\in L$, $\tilde{\alpha}$ denote the constant mapping: $X\times X\to L$ defined by $\tilde{\alpha}(x,y)=\alpha$ for all $x,y\in X$ [12]. For each pair (x,y) of elements x, y of X, the mapping $(x,y)^\bullet : L^{X\times X}\to L$ defined by $(x,y)^\bullet(u)=u(x,y)$ for all $u\in L^{X\times X}$ is a homogeneous fuzzy filter on $X\times X$. Let $\mathcal{U}$ and $\mathcal{V}$ be fuzzy filters on $X\times X$ such that $(x,y)^\bullet\le\mathcal{U}$ and $(y,z)^\bullet\le\mathcal{V}$ hold for some $x,y,z\in X$. Then the *composition* $\mathcal{V}\circ\mathcal{U}$ of $\mathcal{U}$ and $\mathcal{V}$ is [12] the fuzzy filter on $X\times X$ defined by

$$(\mathcal{V}\circ\mathcal{U})(w)=\bigvee_{v\circ u\le w}\big(\mathcal{U}(u)\wedge\mathcal{V}(v)\big) \tag{4.1}$$

for all $w \in L^{X\times X}$, where $u, v, v\circ u \in L^{X\times X}$ and

$$\left(v\circ u\right)\left(x,y\right)=\bigvee_{z\in X}\left(u\left(x,z\right)\wedge v\left(z,y\right)\right) \tag{4.2}$$

for all $x, y\in X$.

By a *fuzzy uniform structure* $\mathcal{U}$ on a set X [12] we mean a fuzzy filter on $X\times X$ such that:

(U1) $\left(x,x\right)^{\bullet}\le \mathcal{U}$ for all $x\in X$.

(U2) $\mathcal{U}=\mathcal{U}^{-1}$.

(U3) $\mathcal{U}\circ\mathcal{U}\le\mathcal{U}$.

A set X equipped with a fuzzy uniform structure $\mathcal{U}$ is called a *fuzzy uniform space.* For any complete chain we have the following result.

Lemma 4.1. *The supremum of two fuzzy uniform structures is a fuzzy uniform structure.*

Proof. Clear.

Proposition 4.2 [12]. *There is a one-to-one correspondence between the fuzzy uniform structures $\mathcal{U}$ on X and the families $\left(\mathcal{U}_\alpha\right)_{\alpha\in L_0}$ of prefilters on $X\times X$ which fulfill the following conditions:*

(u1) $0<\beta\le\alpha$ *implies* $\mathcal{U}_\alpha\subseteq\mathcal{U}_\beta$.

(u2) *For each* $\alpha\in L_0$ *with* $\bigvee_{0<\beta<\alpha}\beta=\alpha$, *we have* $\mathcal{U}_\alpha=\bigcap_{0<\beta<\alpha}\mathcal{U}_\beta$.

(u3) *For all* $\alpha\in L_0$, $u\in\mathcal{U}_\alpha$ *and* $x\in X$, *we have* $\alpha\le u\left(x,x\right)$.

(u4) $u\in\mathcal{U}_\alpha$ *implies* $u^{-1}\in\mathcal{U}_\alpha$ *for all* $\alpha\in L_0$.

(u5) *For each* $\alpha\in L_0$ *and each* $u\in\mathcal{U}_\alpha$, *we have* $\alpha\le\bigvee_{v\in\mathcal{U}_\beta,\, v\circ v\le u}\beta$.

This correspondence is given by $\mathcal{U}_\alpha=\alpha-\mathrm{pr}\mathcal{U}$ *for all* $\alpha\in L_0$ *and* $\mathcal{U}\left(u\right)=\bigvee_{v\in\mathcal{U}_\alpha,\, v\le u}\alpha$ *for all* $u\in L^{X\times X}$. *Now we shall prove the following important results in which those conditions (e1) – (e5) for the family* $\left(\alpha-\mathrm{pr}\mathcal{N}\left(e\right)\right)_{\alpha\in L_0}$ *are necessary to construct fuzzy uniform structures by which the stratified* L*-topological group* $\left(G,\tau\right)$ *is uniformizable. First, we construct these fuzzy uniform structures and then, in another proposition, we show that* $\left(G,\tau\right)$ *is uniformizable.*

Proposition 4.3. *Let* $\left(G,\tau\right)$ *be an* L*-topological group. Then the families* $\left(\mathcal{U}^l_\alpha\right)_{\alpha\in L_0}$ *and* $\left(\mathcal{U}^r_\alpha\right)_{\alpha\in L_0}$ *of the subsets* $\mathcal{U}^l_\alpha$ *and* $\mathcal{U}^r_\alpha$ *of* $L^{G\times G}$ *defined by*

$$\mathcal{U}^l_\alpha=\left\{u\in L^{G\times G}\,\middle|\,u\left(x,y\right)=\left(f\wedge f^{-1}\right)\left(x^{-1}y\right)\text{ for some } f\in\alpha-\mathrm{pr}\mathcal{N}\left(e\right)\right\} \tag{4.3}$$

and

$$\mathcal{U}_{\alpha}^{r}=\left\{u\in L^{G\times G}\,\middle|\,u(x,y)=\left(f\wedge f^{-1}\right)\left(xy^{-1}\right)\text{ for some } f\in\alpha-\mathrm{pr}\mathcal{N}(e)\right\} \tag{4.4}$$

correspond fuzzy uniform structures $\mathcal{U}^{l}$ and $\mathcal{U}^{r}$ on G, respectively by the following:

$$\mathcal{U}_{\alpha}^{l}=\alpha-\mathrm{pr}\mathcal{U}^{l}\ \textit{ and }\ \mathcal{U}^{l}(u)=\bigvee_{v\in\mathcal{U}_{\alpha}^{l},v\le u}\alpha \tag{4.5}$$

and

$$\mathcal{U}_{\alpha}^{r}=\alpha-\mathrm{pr}\mathcal{U}^{r}\ \textit{ and }\ \mathcal{U}^{r}(u)=\bigvee_{v\in\mathcal{U}_{\alpha}^{r},v\le u}\alpha \tag{4.6}$$

Proof. Since $\tilde{0}(x,x)=0\neq 1=\left(f\wedge f^{-1}\right)(e)=\left(f\wedge f^{-1}\right)\left(x^{-1}x\right)$ for all $f\in\alpha-\mathrm{pr}\mathcal{N}(e)$ and all $x\in G$, then $\tilde{0}\notin\mathcal{U}_{\alpha}^{l}$ for all $\alpha\in L_{0}$. Also, $\tilde{1}\in\mathcal{U}_{\alpha}^{l}$ for all $\alpha\in L_{0}$, from that there exists a symmetric fuzzy set $f=e_{1}=\left(x^{-1}yy^{-1}x\right)_{1}=\left(x^{-1}y\right)_{1}\left(y^{-1}x\right)_{1}\in\alpha-\mathrm{pr}\mathcal{N}(e)$ such that $\left(f\wedge f^{-1}\right)\left(x^{-1}y\right)=f\left(x^{-1}y\right)\wedge f\left(y^{-1}x\right)=1$ for all $x,y\in G$.

Let $u\in\mathcal{U}_{\alpha}^{l}$ for all $\alpha\in L_{0}$ and $v\geq u$. Then $v(x,y)\geq\left(f\wedge f^{-1}\right)\left(x^{-1}y\right)$ for some $f\in\alpha$ $-\mathrm{pr}\mathcal{N}(e)$ and for all $x,y\in G$. But $v\leq\tilde{1}\in\mathcal{U}_{\alpha}^{l}$ implies that there is $g\in\alpha-\mathrm{pr}\mathcal{N}(e)$ such that $v(x,y)\leq\left(g\wedge g^{-1}\right)\left(x^{-1}y\right)$ for all $x,y\in G$. That is, there is some $h\in\alpha-\mathrm{pr}\mathcal{N}(e)$ such that $v(x,y)=\left(h\wedge h^{-1}\right)\left(x^{-1}y\right)$ for all $x,y\in G$. Hence $v\in\mathcal{U}_{\alpha}^{l}$ for all $\alpha\in L_{0}$. Since $(f\wedge g)\in\alpha-\mathrm{pr}\mathcal{N}(e)$ whenever $f\in\alpha-\mathrm{pr}\mathcal{N}(e)$ and $g\in\alpha-\mathrm{pr}\mathcal{N}(e)$, then for any $u,v\in\mathcal{U}_{\alpha}^{l}$, we get that

$$\begin{aligned}(u\wedge v)(x,y)&=u(x,y)\wedge v(x,y)\\&=\left(f\wedge f^{-1}\right)\left(x^{-1}y\right)\wedge\left(g\wedge g^{-1}\right)\left(x^{-1}y\right)\text{ for some } f,g\in\alpha-\mathrm{pr}\mathcal{N}(e)\\&=\left((f\wedge g)\wedge(f\wedge g)^{-1}\right)\left(x^{-1}y\right)\text{ for some } f,g\in\alpha-\mathrm{pr}\mathcal{N}(e).\end{aligned}$$

Hence $(u\wedge v)\in\mathcal{U}_{\alpha}^{l}$ for all $\alpha\in L_{0}$. Thus $\mathcal{U}_{\alpha}^{l}$ is a prefilter on $G\times G$ for all $\alpha\in L_{0}$.

Now, let $0<\beta\leq\alpha$ and $u\in\mathcal{U}_{\alpha}^{l}$. Then from (e1) for the family $\left(\alpha-\mathrm{pr}\mathcal{N}(e)\right)_{\alpha\in L_{0}}$, we get that $u(x,y)=\left(g\wedge g^{-1}\right)\left(x^{-1}y\right)$ for some $g\in\beta-\mathrm{pr}\mathcal{N}(e)$ for all $x,y\in G$, and then $u\in\mathcal{U}_{\beta}^{l}$. Hence, the condition (u1) of Proposition 4.2 holds. From (u1) of Proposition 4.2 and from (e2) for $\left(\alpha-\mathrm{pr}\mathcal{N}(e)\right)_{\alpha\in L_{0}}$. We get that (u2) of Proposition 4.2 is fulfilled. From (e3(and (e4) for $\left(\alpha-\mathrm{pr}\mathcal{N}(e)\right)_{\alpha\in L_{0}}$, we have for all $\alpha\in L_{0}$ and all $u\in\mathcal{U}_{\alpha}^{l}$ that

$$u(x,x) = \left(f \wedge f^{-1}\right)\left(x^{-1}x\right) = \left(f \wedge f^{-1}\right)(e) \geq \alpha$$

for some $f \in \alpha - \mathrm{pr}\mathcal{N}(e)$. Hence, (u3) of Proposition 4.2 holds.

For all $\alpha \in L_0$ and all $u \in \mathcal{U}_\alpha^l$, we have for all $x, y \in G$ that

$$u^{-1}(x,y) = u(y,x) = \left(f \wedge f^{-1}\right)\left(y^{-1}x\right)$$

for some $f \in \alpha - \mathrm{pr}\mathcal{N}(e)$. Since (3.4) implies, for all $x, y \in G$, that

$$\left(f \wedge f^{-1}\right)\left(x^{-1}y\right) = f\left(x^{-1}y\right) \wedge f^{-1}\left(x^{-1}y\right) = f^{-1}\left(y^{-1}x\right) \wedge f\left(y^{-1}x\right) = \left(f \wedge f^{-1}\right)\left(y^{-1}x\right)$$

that is, $u(x,y) = u(y,x)$ for all $x, y \in G$, then $u \in \mathcal{U}_\alpha^l$ if and only if $u^{-1} \in \mathcal{U}_\alpha^l$ and thus (u4) of Proposition 4.2 holds.

From (e5) for $\left(\alpha - \mathrm{pr}\mathcal{N}(e)\right)_{\alpha \in L_0}$, we have for all $\alpha \in L_0$ and all $f \in \alpha - \mathrm{pr}\mathcal{N}(e)$ that there exists $g \in \beta - \mathrm{pr}\mathcal{N}(e)$, $\beta \in L_0$, such that $gg \leq f$. For any $u \in \mathcal{U}_\alpha^l$ and all $x, y \in G$, we have $u(x,y) = \left(f \wedge f^{-1}\right)\left(x^{-1}y\right)$ for some $f \in \alpha - \mathrm{pr}\mathcal{N}(e)$, which means that there exists $v \in \mathcal{U}_\beta^l$, $\beta \in L_0$, such that (4.2) implies for all $x, y \in G$ that:

$$(v \circ v)(x,y) = \bigvee_{z \in G}\left(v(x,z) \wedge v(z,y)\right) = \bigvee_{z \in G}\left(\left(g \wedge g^{-1}\right)\left(x^{-1}z\right) \wedge \left(g \wedge g^{-1}\right)\left(z^{-1}y\right)\right)$$
$$\leq \left(f \wedge f^{-1}\right)\left(x^{-1}y\right) = u(x,y).$$

Hence, by means of (e5) for $\left(\alpha - \mathrm{pr}\mathcal{N}(e)\right)_{\alpha \in L_0}$, we get

$$\alpha \leq \bigvee_{v \in \mathcal{U}_\beta^l, (v \circ v) \leq u} \beta = \bigvee_{g \in \beta - \mathrm{pr}\mathcal{N}(e), gg \leq f} \beta$$

and then (u5) of Proposition 4.2 holds.

Now, we have the family $\left(\mathcal{U}_\alpha^l\right)_{\alpha \in L_0}$ is a family of prefilters on $G \times G$ and fulfills the conditions (u1) – (u5). Form Proposition 4.2, we get that $\left(\mathcal{U}_\alpha^l\right)_{\alpha \in L_0}$ corresponds a fuzzy uniform structure $\mathcal{U}^l$ on G. This correspondence is given by

$$\mathcal{U}^l(u) = \bigvee_{v \in \mathcal{U}_\alpha^l, v \leq u} \alpha \text{ and } \mathcal{U}_\alpha^l = \alpha - \mathrm{pr}\mathcal{U}^l.$$

The same proof can be done with the family $\left(\mathcal{U}_\alpha^r\right)_{\alpha \in L_0}$.

Definition 4.1. $\mathcal{U}^l$ and $\mathcal{U}^r$ defined by (4.5) and (4.6) are called *left* fuzzy uniform structure and *right* fuzzy uniform structure on G, respectively.

An L -topological group (G,τ) is called *abelian* if the group G is abelian.

Proposition 4.4. *For abelian L -topological groups, the left and the right fuzzy uniform structures coincide.*

Proof. Since

$$\left(f \wedge f^{-1}\right)\left(x^{-1}y\right) = \left(f \wedge f^{-1}\right)\left(y^{-1}x\right) = \left(f \wedge f^{-1}\right)\left(xy^{-1}\right)$$

for all $x, y \in G$ and for some $f \in \alpha - \mathrm{pr}\mathcal{N}(e)$, then $\mathcal{U}_\alpha^l = \mathcal{U}_\alpha^r$ for all $\alpha \in L_0$. Therefore, $\mathcal{U}^l = \mathcal{U}^r$.

Let $\mathcal{U}$ be a fuzzy filter on $X \times X$ such that $(x,x)^\bullet \le \mathcal{U}$ holds for all $x \in X$, and let $\mathcal{M}$ be a fuzzy filter on X. Then the mapping $\mathcal{U}[\mathcal{M}]: L^X \to L$, defined by

$$\mathcal{U}[\mathcal{M}](f) = \bigvee_{u[g] \le f} \left(\mathcal{U}(u) \wedge \mathcal{M}(g)\right) \tag{4.7}$$

for all $f \in L^X$, is a fuzzy filter on X, called the image of $\mathcal{M}$ with respect to $\mathcal{U}$ [12], where $u \in L^{X\times X}$ and $g, u[g] \in L^X$ such that:

$$u[g](x) = \bigvee_{y \in X} \left(g(y) \wedge u(y,x)\right). \tag{4.8}$$

Proposition 4.5 [12]. *Let $\mathcal{U}$ be a fuzzy filter on $X \times X$ such that $(x,x)^\bullet \le \mathcal{U}$ holds for all $x \in X$, and let $\mathcal{M}$ be a fuzzy filter on X. Then the family $(\mathcal{L}_\alpha)_{\alpha \in L_0}$ with*

$$\mathcal{L}_\alpha = \left\{ f \in L^X \,\middle|\, u[g] \le f \text{ for some } u \in \alpha - \mathrm{pr}\mathcal{U} \text{ and } g \in \alpha - \mathrm{pr}\mathcal{M} \right\}$$

is a valued fuzzy filter base of $\mathcal{U}[\mathcal{M}]$, which consists of prefilters on X such that $\alpha \le \beta$ implies $\mathcal{L}_\alpha \supseteq \mathcal{L}_\beta$ for all $\alpha, \beta \in L_0$.

Remark 4.1. From Proposition 4.5, we get for a fuzzy uniform structure $\mathcal{U}$ on X and a homogeneous fuzzy filter $\dot{x}$ at $x \in X$, that the family $(\mathcal{L}_\alpha)_{\alpha \in L_0}$ with

$$\mathcal{L}_\alpha = \left\{ f \in L^X \,\middle|\, u[g] \le f \text{ for some } u \in \alpha - \mathrm{pr}\mathcal{U} \text{ and } \alpha \le g(x) \right\} \tag{4.9}$$

is a valued fuzzy filter base of $\mathcal{U}[\dot{x}]$, and moreover $\mathcal{L}_\alpha = \alpha - \mathrm{pr}\mathcal{U}[\dot{x}]$ for all $\alpha \in L_0$.

To each fuzzy uniform structure $\mathcal{U}$ on X is associated a stratified fuzzy topology $\tau_\mathcal{U}$. The related interior operator $\mathrm{int}_\mathcal{U}$ is given by [12].

$$\left(\operatorname{int}_{\mathcal{U}} f\right)(x) = \mathcal{U}[\dot{x}](f) \tag{4.10}$$

for all $x \in X$, $f \in L^X$. A fuzzy set $f \in L^X$ is called a $\tau_{\mathcal{U}}$*-neighborhood* of $x \in X$ provided $\alpha \leq \operatorname{int}_{\mathcal{U}} f(x)$ for some $\alpha \in L_0$.

In the following proposition, we show that every stratified L-topological group is uniformizable.

Proposition 4.6. *Any stratified L-topological group (G,τ) is uniformizable. That is, $\tau_{\mathcal{U}^l} = \tau_{\mathcal{U}^r} = \tau_{(\mathcal{U}^l \vee \mathcal{U}^r)} = \tau$.*

Proof. From Lemma 4.1 and Proposition 4.3, we get that both of $\mathcal{U}^l$, $\mathcal{U}^r$ and $\mathcal{U}^l \vee \mathcal{U}^r$ are fuzzy uniform structures on G.

Since for all $x \in G$ and all $f \in L^G$ we have, from (4.7), (4.10) and Remark 4.1, that:

$$\operatorname{int}_{\mathcal{U}^l} f(x) = \mathcal{U}^l[\dot{x}](f) = \bigvee_{u[g] \leq f} \left(\mathcal{U}^l(u) \wedge g(x)\right) = 1$$

is equivalent to

$$\operatorname{int}_{\mathcal{U}^r} f(x) = \mathcal{U}^r[\dot{x}](f) = \bigvee_{u[g] \leq f} \left(\mathcal{U}^r(u) \wedge g(x)\right) = 1$$

equivalent to

$$\operatorname{int}_{(\mathcal{U}^l \vee \mathcal{U}^r)} f(x) = \left(\mathcal{U}^l \vee \mathcal{U}^r\right)[\dot{x}](f) = \bigvee_{u[g] \leq f} \left(\left(\mathcal{U}^l \vee \mathcal{U}^r\right)(u) \wedge g(x)\right) = 1,$$

which means that f is a $\tau_{\mathcal{U}^l}$-neighborhood of an element x if and only if it is a $\tau_{\mathcal{U}^r}$-neighborhood of x if and only if it is a $\tau_{(\mathcal{U}^l \vee \mathcal{U}^r)}$-neighborhood of x. Hence $\tau_{\mathcal{U}^l} = \tau_{\mathcal{U}^r} = \tau_{(\mathcal{U}^l \vee \mathcal{U}^r)}$.

From (4.7) and (4.8), and also from Remark 4.1, we have

$$\mathcal{U}^l[\dot{x}](f) = \bigvee_{g \in \alpha - \operatorname{pr}\mathcal{U}^l[\dot{x}], g \leq f} \alpha = \bigvee_{u[g] \leq f} \left(\mathcal{U}^l(u) \wedge g(x)\right) = \bigvee_{h \in \alpha - \operatorname{pr}\mathcal{N}(x), h \leq f} \alpha = \mathcal{N}(x)(f)$$

for all $x \in G$ and all $f \in L^G$. Hence, the fuzzy neighborhood filter $\mathcal{U}^l[\dot{x}]$ of $(G,\tau_{\mathcal{U}^l})$ at every $x \in G$ is identical with the fuzzy neighborhood filter $\mathcal{N}(x)$ at every x in the L-topological group (G,τ). Thus, $\tau_{\mathcal{U}^l} = \tau$, and therefore (G,τ) is uniformizable.

In the following we show that these conditions (e1) – (e5) for a family of prefilters on G are also sufficient to construct form the group G a stratified L-topological group.

Proposition 4.7. *Let G be a group and e the identity element of G, and let $\left(\mathcal{V}_\alpha^e\right)_{\alpha \in L_0}$ be a family of prefilters on G fulfilling conditions (e1) – (e5). Defining, for each $\alpha \in L_0$, the subsets*

$$\mathcal{U}_\alpha^l = \left\{ u \in L^{G\times G} \,\middle|\, u(x,y) = \left(f \wedge f^{-1}\right)\left(x^{-1}y\right) \text{ for some } f \in \mathcal{V}_\alpha^e \right\}$$

and

$$\mathcal{U}_\alpha^r = \left\{ u \in L^{G\times G} \,\middle|\, u(x,y) = \left(f \wedge f^{-1}\right)\left(xy^{-1}\right) \text{ for some } f \in \mathcal{V}_\alpha^e \right\}$$

of $L^{G\times G}$. *Hence, we have the left and the right fuzzy uniform structures* $\mathcal{U}^l$ *and* $\mathcal{U}^r$ *on* G *defined by (4.5) and (4.6), respectively. Moreover,* $\tau_{\mathcal{U}^l} = \tau_{\mathcal{U}^r} = \tau_{(\mathcal{U}^l \vee \mathcal{U}^r)}$ *is a stratified fuzzy topology* τ *on* G *for which the pair* (G,τ) *is a stratified* L*-topological group. Finally, for each* $\alpha \in L_0$, *we have* $\mathcal{V}_\alpha^e = \alpha - \text{pr}\mathcal{N}(e)$, *where* $\mathcal{N}(e)$ *is the fuzzy neighborhood filter at* e *with respect to the fuzzy topology* τ *on* G.

Proof. As in Proposition 4.3 and 4.6, we get that $\mathcal{U}^l$ and $\mathcal{U}^r$ are the left and the right fuzzy uniform structures on G for which $\tau_{\mathcal{U}^l} = \tau_{\mathcal{U}^r} = \tau_{(\mathcal{U}^l \vee \mathcal{U}^r)}$ is a fuzzy topology on the group G. Denote $\tau_{\mathcal{U}^l} = \tau_{\mathcal{U}^r} = \tau_{(\mathcal{U}^l \vee \mathcal{U}^r)}$ by τ. It remains to prove that (G,τ) is an L-topological group and that $\mathcal{V}_\alpha^e = \alpha - \text{pr}\mathcal{N}(e)$ for all $\alpha \in L_0$.

Now, from that the conditions of proposition 2.1 are equivalent to the conditions (e1) – (e2), we get that

$$\mathcal{V}_\alpha^e = \alpha - \text{pr}\mathcal{U}^l[\dot{e}] = \alpha - \text{pr}\mathcal{U}^r[\dot{e}] = \alpha - \text{pr}\left(\mathcal{U}^l \vee \mathcal{U}^r\right)[\dot{e}]$$

for all $\alpha \in L_0$. That is, $\mathcal{V}_\alpha^e = \alpha - \text{pr}\mathcal{N}(e)$ for all $\alpha \in L_0$, where $\mathcal{N}(e)$ is the fuzzy neighborhood filter of (G,τ) at e.

From conditions (e4) and (e5) of the prefilters $\alpha - \text{pr}\mathcal{N}(e)$ for all $\alpha \in L_0$, we get that for all $f \in \alpha - \text{pr}\mathcal{N}(e)$, there exist $g,h \in \alpha - \text{pr}\mathcal{N}(e)$ for some $\alpha \in L_0$ such that $g^{-1}h \le f$, which means that

$$(ga)^{-1}(hb) = a^{-1}\left(g^{-1}h\right)b \le a^{-1}fb\,.$$

That is, from Lemma 3.2, we get that for all $\lambda = a^{-1}fb \in \alpha - \text{pr}\mathcal{N}\left(a^{-1}b\right)$, there exist $\mu = ga \in \alpha - \text{pr}\mathcal{N}(a)$ and $\nu = hb \in \alpha - \text{pr}\mathcal{N}(b)$ such that $\mu^{-1}\nu \le \lambda$. Hence, (G,τ) is an L-topological group. Let us define the following.

Definition 4.2. Let $\mathcal{U}$ be a fuzzy uniform structure on a set X. Then

(1) $u \in L^{X\times X}$ is called a *surrounding* provided $\mathcal{U}(u) \ge \alpha$ for some $\alpha \in L_0$ and $u = u^{-1}$,

(2) A surrounding $u \in L^{X\times X}$ is called *left* (*right*) *invariant* provided $u(ax,ay) = u(x,y)$ $\left(u(xa,ya) = u(x,y)\right)$ for all $a,x,y \in X$,

(3) $\mathcal{U}$ is called a *left* (*right*) *invariant* fuzzy uniform structure if $\mathcal{U}$ has a valued fuzzy filter base consists of left (right) invariant surroundings.

Now, from Proposition 4.3, we have this remark.

Remark 4.2. In the L-topological group (G,τ), for each element u in $\mathcal{U}_\alpha^l$, defined by (4.3), we have $\mathcal{U}_\alpha^l(u) \geq \alpha$ for some $\alpha \in L_0$ and also, for all $x,y \in G$ and each $u \in \mathcal{U}_\alpha^l$, we have

$$\begin{aligned} u(x,y) &= \left(f \wedge f^{-1}\right)\left(x^{-1}y\right) \text{ for some } f \in \alpha - \mathrm{pr}\mathcal{N}(e) \\ &= \left(f \wedge f^{-1}\right)\left(y^{-1}x\right) \text{ for some } f \in \alpha - \mathrm{pr}\mathcal{N}(e) \\ &= u(y,x) = u^{-1}(x,y). \end{aligned}$$

That is, $\mathcal{U}_\alpha^l$ is a prefilter of surroundings. Also, for all $a,x,y \in G$, we have

$$\begin{aligned} u(ax,ay) &= \left(f \wedge f^{-1}\right)\left((ax)^{-1}(ay)\right) \text{ for some } f \in \alpha - \mathrm{pr}\mathcal{N}(e) \\ &= \left(f \wedge f^{-1}\right)\left(x^{-1}y\right) \text{ for some } f \in \alpha - \mathrm{pr}\mathcal{N}(e) \\ &= u(x,y) \text{ for all } u \in \mathcal{U}_\alpha^l \text{ and for all } x,y \in G. \end{aligned}$$

Thus, the elements of $\mathcal{U}_\alpha^l$ are left invariant surroundings. Moreover, $\left(\mathcal{U}_\alpha^l\right)_{\alpha \in L_0}$ is a valued fuzzy filter base for the left fuzzy uniform structure $\mathcal{U}^l$ defined by (4.5), and hence $\mathcal{U}^l$ is a left invariant fuzzy uniform structure on G. By the same way, $\mathcal{U}^r$, defined by (4.6), is a right invariant fuzzy uniform structure on G.

Notice that: Between any two systems of sets $\mathcal{A}$ and $\mathcal{B}$, we recall that $\mathcal{A}$ is called *coarser than* $\mathcal{B}$ if for any $A \in \mathcal{A}$, there is $B \in \mathcal{B}$ such that $B \subseteq A$.

The following important proposition is now obtained from our last results.

Proposition 4.8. *Let (G,τ) be a stratified L-topological group. Then there exist on G a unique left invariant fuzzy uniform structure $\mathcal{U}^l$ and a unique right invariant fuzzy uniform structure $\mathcal{U}^r$ compatible with τ, constructed in Proposition 4.3 using the family $\left(\alpha - \mathrm{pr}\mathcal{N}(e)\right)_{\alpha \in L_0}$ of all prefilters $\alpha - \mathrm{pr}\mathcal{N}(e)$, where $\mathcal{N}(e)$ is the fuzzy neighborhood filter at the identity element e of the L-topological group (G,τ).*

Proof. From Propositions 4.3 and 4.6, and Remark 4.2, we have $\mathcal{U}^l$ and $\mathcal{U}^r$ are the left and the right invariant fuzzy uniform structures on G, respectively for which $\tau_{\mathcal{U}^l} = \tau_{\mathcal{U}^r} = \tau$. Suppose that $\left(\mathcal{V}_\alpha^l\right)_{\alpha \in L_0}$ is a valued fuzzy filter base for a left invariant fuzzy uniform structure $\mathcal{V}^l$ on G such that $\tau_{\mathcal{V}^l} = \tau_{\mathcal{U}^l} = \tau$.

Since for any $v_1 \in \mathcal{V}_\alpha^l$, there exists $v_2 \in \mathcal{V}_\alpha^l$ with $v_2 \le v_1$ and $v_2(ax,ay) = v_2(x,y)$ for all $a,x,y \in G$. From (4.8), we get that $v_2[e_1](x) = v_2(e,x)$ for all $x \in G$, that is, $v_2[e_1](e) = v_2(e,e) \ge \alpha$ and there exists a left invariant surrounding $u \in \mathcal{U}_\alpha^l$ such that $u[e_1] \le v_2[e_1]$. Now, $u(x,y) = u(xx^{-1},x^{-1}y) = u(e,x^{-1}y) = u[e_1](x^{-1}y) \le v_2[e_1](x^{-1}y)$ for all $x,y \in G$, which means that $u(x,y) = v_2(e,x^{-1}y) = v_2(x,y)$ and also we have $v_2 \le v_1$, so $u \le v_1$. That is, for all $\alpha \in L_0$ and for any $v_1 \in \mathcal{V}_\alpha^l$, there exists $u \in \mathcal{U}_\alpha^l$ such that $u \le v_1$, and this means that $\mathcal{V}_\alpha^l$ is coarser than $\mathcal{U}_\alpha^l$ for all $\alpha \in L_0$. By the same way, we can show that $\mathcal{U}_\alpha^l$ is coarser than $\mathcal{V}_\alpha^l$ for all $\alpha \in L_0$, and thus $\mathcal{V}_\alpha^l = \mathcal{U}_\alpha^l$ for all $\alpha \in L_0$. Hence, $\mathcal{V}^l = \mathcal{U}^l$.

Similarly, one can prove that the right invariant fuzzy uniform structure $\mathcal{U}^r$ is unique.

5. The relation between the L-topological groups and the $GT_{3\frac{1}{2}}$-spaces

In this section we shall show and prove the relation between our notion of $GT_{3\frac{1}{2}}$-spaces and the notion of L-topological groups defined in [1]. In [2,3,5] we had defined the fuzzy separation axioms GT_i, $i = 0,1,2,3,3_{\frac{1}{2}},4$. Here, we recall some of these axioms which we need in the following.

A fuzzy topological space (X,τ) is called [2,3,5]:

(1) GT_0 if for all $x,y \in X$ with $x \ne y$ we have $\dot{x} \not\le \mathcal{N}(y)$ or $\dot{y} \not\le \mathcal{N}(x)$.

(2) GT_1 if for all $x,y \in X$ with $x \ne y$ we have $\dot{x} \not\le \mathcal{N}(y)$ and $\dot{y} \not\le \mathcal{N}(x)$.

(3) GT_2 if for all $x,y \in X$ with $x \ne y$ we have $\mathcal{N}(x) \wedge \mathcal{N}(y)$ does not exist.

(4) GT_3 if it is GT_1 and if for all $x \in X$ and all $F \in \tau'$ with $x \notin F$, we have $\mathcal{N}(x) \wedge \mathcal{N}(F)$ does not exist.

(5) *completely regular* if for all $x \notin F \in \tau'$, there exists a fuzzy continuous mapping $f:(X,\tau) \to (I_L,\Im)$ such that $f(x) = \overline{1}$ and $f(y) = \overline{0}$ for all $y \in F$.

(6) $GT_{3\frac{1}{2}}$ (or L-Tychonoff) if it is GT_1 and completely regular.

(7) GT_4 if it is GT_1 and if for all $F,G \in \tau'$ with $F \cap G = \emptyset$, we have $\mathcal{N}(F) \wedge \mathcal{N}(G)$ does not exist.

Denote by GT_i-*space* the fuzzy topological space which is GT_i, $i = 0,1,2,3,3_{\frac{1}{2}},4$.

Proposition 5.1 [2,3,5]. *Every GT_i-space is GT_{i-1}-space for each $i = 1,2,3,4$, and $GT_{3\frac{1}{2}}$-spaces fulfill the following: every GT_4-space is a $GT_{3\frac{1}{2}}$-space and every $GT_{3\frac{1}{2}}$-space is a GT_3-space.*

Proposition 5.2 [6]. *If $\mathcal{U}$ is a fuzzy uniform structure on a set X and $\tau_{\mathcal{U}}$ the fuzzy topology associated to $\mathcal{U}$, then $(X,\tau_{\mathcal{U}})$ is a completely regular space. The fact that the fuzzy topology of an L-topological group can be induced by a left or right invariant fuzzy uniform structure leads us to our fundamental results in this section as follows.*

Proposition 5.3. *The fuzzy topology of an L-topological group is completely regular.*

Proof. The proof goes directly from Proposition 4.6 and 5.2.

Definition 5.1. An L-topological group (G,τ) is called *separated* if for the identity element e, we have $\bigwedge_{f\in\alpha-\mathrm{pr}\mathcal{N}(e)} f(e)\geq\alpha$ and $\bigwedge_{f\in\alpha-\mathrm{pr}\mathcal{N}(e)} f(x)<\alpha$ for all $x\in G$ with $x\neq e$ and for all $\alpha\in L_0$. A fuzzy uniform structure $\mathcal{U}$ on a set X is called *separated* [4] if for all $x,y\in X$ with $x\neq y$ there is $u\in L^{X\times X}$ such that $\mathcal{U}(u)=1$ and $u(x,y)=0$. The space $(X,\mathcal{U})$ is called *separated fuzzy uniform space.*

Proposition 5.4 [4]. *Let X be a set, $\mathcal{U}$ a fuzzy uniform structure on X and $\tau_{\mathcal{U}}$ the fuzzy topology associated with $\mathcal{U}$. Then $(X,\mathcal{U})$ is separated if and only if $(X,\tau_{\mathcal{U}})$ is GT_0-space. In the following result we have shown the expected relation between our notion of $GT_{3\frac{1}{2}}$-spaces and the notion of L-topological groups.*

Proposition 5.5. *Let (G,τ) be an L-topological group. Then the following statements are equivalent.*

(1) *The fuzzy topology τ is GT_0.*

(2) *The fuzzy topology τ is GT_1.*

(3) *The fuzzy topology τ is GT_2.*

(4) *The fuzzy topology τ is $GT_{3\frac{1}{2}}$.*

(5) *$\mathcal{U}^l$ is separated.*

(6) *$\mathcal{U}^r$ is separated.*

(7) *The L-topological group (G,τ) is separated.*

Proof. (1) $\Rightarrow$ (2) Let $x\neq y$ in G, then for one point (say x) there exists a τ-neighborhood f such that $\mathrm{int}_\tau f(x)\geq\alpha>f(y)$, which means that there is $h\in\alpha-\mathrm{pr}\mathcal{N}(e)$ such that $h=x^{-1}f$ and then $k=h\wedge h^{-1}$ is a symmetric τ-neighborhood of e, and this means that the fuzzy set $g=yk$ is a τ-neighborhood of y for which $\mathrm{int}_\tau g(y)\geq\alpha>g(x)$ because if otherwise $g(x)=yk(x)\geq\alpha$, then

$$\alpha\leq g^{-1}\left(x^{-1}\right)=\left(h\wedge h^{-1}\right)y^{-1}\left(x^{-1}\right)=\left(x^{-1}f\wedge f^{-1}x\right)y^{-1}\left(x^{-1}\right)\leq x^{-1}fy^{-1}\left(x^{-1}\right),$$

that is, $fy^{-1}(e) \geq \alpha$, and then $f(y) \geq \alpha$ which is a contradiction. Hence there exists a τ-neighborhood g of y such that $\text{int}_\tau\, g(y) \geq \alpha > g(x)$, and thus (G,τ) is a GT_1-space.

(2) $\Rightarrow$ (3) It is clear from Proposition 5.1 and 5.3.

(3) $\Rightarrow$ (4) Obvious.

(4) $\Rightarrow$ (5) and (4) $\Rightarrow$ (6) The proof comes from Proposition 4.6, and from Proposition 5.1 and 5.4.

(5) $\Rightarrow$ (7) Since $\mathcal{U}^l$ is separated then, by means of Proposition 4.6 and 5.4, $\tau = \tau_{\mathcal{U}^l}$ is GT_0. Thus for any $x \neq e$ in G, there exists $f \in \alpha - \text{pr}\mathcal{N}(e)$ such that $f(x) < \alpha \leq \text{int}_\tau\, f(e) \leq f(e)$. Hence, $\bigwedge_{f\in\alpha-\text{pr}\mathcal{N}(e)} f(x) \geq \alpha$ whenever $x = e$ and $\bigwedge_{f\in\alpha-\text{pr}\mathcal{N}(e)} f(x) < \alpha$ otherwise. That is, (G,τ) is a separated L-topological group.

(6) $\Rightarrow$ (7) The proof goes similar to the case (5) $\Rightarrow$ (7).

(7) $\Rightarrow$ (1) If $x, y \in G$ with $x \neq y$, then $x^{-1}y \neq e$ and then $\bigwedge_{f\in\alpha-\text{pr}\mathcal{N}(e)} f\left(x^{-1}y\right) < \alpha$, which means that there exists $f \in \alpha - \text{pr}\mathcal{N}(e)$ such that $f\left(x^{-1}y\right) < \alpha$, that is, $xf(y) = \bigwedge_{f(z)>0} (xz)_1(y) < \alpha$, where $z = x^{-1}y$ is not allowed. Since $\left\{xf \,\middle|\, f \in \alpha - \text{pr}\mathcal{N}(e)\right\}$ is itself $\alpha - \text{pr}\mathcal{N}(x)$, that is, the set of all α-fuzzy neighborhoods of x and $xf(y) < \alpha$. Hence, $xf(y) < \alpha \leq \text{int}_\tau (xf)(x)$. Thus, (G,τ) is GT_0.

References

[1] T. M. G. Ahsanullah, On fuzzy topological groups and semigroups, Ph. D Thesis, Faculty of science, Free university of Brussels, (1984).

[2] F. Bayoumi and I. Ibedou, T_i-spaces, I, *The Journal of The Egyptian Mathematical Society*, Vol. 10 (2) (2002), 179-199.

[3] F. Bayoumi and I. Ibedou, T_i-spaces, II, *The Journal of The Egyptian Mathematical Society*, Vol. 10 (2) (2002), 201-215.

[4] F. Bayoumi and I. Ibedou, The relation between the GT_i-spaces and fuzzy proximity spaces, G-compactspaces, fuzzy uniform spaces, *The Journal of Chaos, Solitons and Fractals,* 20 (2004), 955-966.

[5] F. Bayoumi and I. Ibedou, $GT_{3\frac{1}{2}}$-spaces, I, *The journal of the Egyptian mathematical society*, Vol.14(2) (2006), 243-264.

[6] F. Bayoumi and I. Ibedou, $GT_{3\frac{1}{2}}$-spaces, II, *The journal of the Egyptian mathematical society*, Vol.14(2)(2006), 265-282.

[7] F. Bayoumi, On initial and final L-topological groups, *Fuzzy Sets and Systems,* 156 (2005), 43-54.

[8] C. L. Chang, Fuzzy topological spaces, *J. Math. Anal. Appl.,* 24 (1968), 182-190.

[9] P. Eklundand W. Gahler, Fuzzy filter functors and convergence, in Applications of category theory to fuzzy subsets, *Kluwer Academi Publishers, Dorderecht et al.,* (1992), 109-136.

[10] W. Gahler, The general fuzzy filter approach to fuzzy topology, I, *Fuzzy Sets and Systems,* 76 (1995), 205-224.

[11] W. Gahler, The general fuzzy filter approach to fuzzy topology, II, *Fuzzy Sets and Systems*, 76 (1995), 225-246.

[12] W. Gahler, F. Bayoumi, A. Kandil and A. Nouh, The theory of global fuzzy neighborhood structures, (III), Fuzzy uniform structures, *Fuzzy Sets and Systems*, 98 (1998), 175-199.

[13] J. A. Goguen, L -fuzzy sets, *J. Math. Anal. Appl.*, 18 (1967), 145-174.

[14] T. Husain, Introduction to topological groups, Huntington, New York, (1981).

[15] R. Lowen, Convergence inf uzzy topological spaces, *General Topology and Appl.*, k10 (1979), 147-160.